

A COLLECTION OF METRIC MAHLER MEASURES

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ABSTRACT. Let $M(\alpha)$ denote the Mahler measure of the algebraic number α . In a recent paper, Dubickas and Smyth constructed a metric version of the Mahler measure on the multiplicative group of algebraic numbers. Later, Fili and the author used similar techniques to study a non-Archimedean version. We show how to generalize the above constructions in order to associate, to each point in $(0, \infty]$, a metric version M_x of the Mahler measure, each having a triangle inequality of a different strength. We are able to compute $M_x(\alpha)$ for sufficiently small x , identifying, in the process, a function \bar{M} with certain minimality properties. Further, we show that the map $x \mapsto M_x(\alpha)$ defines a continuous function on the positive real numbers.

1. INTRODUCTION

Let f be a polynomial with complex coefficients given by

$$f(z) = a \cdot \prod_{n=1}^N (z - \alpha_n).$$

We define the (*logarithmic*) *Mahler measure* M of f by

$$M(f) = \log |a| + \sum_{n=1}^N \log^+ |\alpha_n|.$$

If α is a non-zero algebraic number, we define the Mahler measure of α by

$$M(\alpha) = M(\min_{\mathbb{Z}}(\alpha)).$$

In other words, $M(\alpha)$ is simply the Mahler measure of the minimal polynomial of α over \mathbb{Z} . It is well-known that

$$(1.1) \quad M(\alpha) = M(\alpha^{-1})$$

for all algebraic numbers α .

It is a consequence of a theorem of Kronecker that $M(\alpha) = 0$ if and only if α is a root of unity. In a famous 1933 paper, D.H. Lehmer [5] asked whether there exists a constant $c > 0$ such that $M(\alpha) \geq c$ in all other cases. He could find no algebraic number with Mahler measure smaller than that of

$$\ell(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

which is approximately $0.16\dots$. Although the best known general lower bound is

$$M(\alpha) \gg \left(\frac{\log \log \deg \alpha}{\log \deg \alpha} \right)^3,$$

2000 *Mathematics Subject Classification.* Primary 11R04; Secondary 26A48.

Key words and phrases. Weil height, Mahler measure, metric Mahler measure, Lehmer's problem.

due to Dobrowolski [2], uniform lower bounds haven been established in many special cases (see [1, 12, 13], for instance). Furthermore, numerical evidence provided by Mossinghoff [6, 7] and Mossinghoff, Pinner and Vaaler [8] suggests there does, in fact, exist such a constant c . This leads to the following conjecture, which we will now call Lehmer's conjecture.

Conjecture (Lehmer's conjecture). There exists a real number $c > 0$ such that if $\alpha \in \overline{\mathbb{Q}}^\times$ is not a root of unity then $M(\alpha) \geq c$.

In an effort to create a geometric structure on the multiplicative group of algebraic numbers $\overline{\mathbb{Q}}^\times$, Dubickas and Smyth [3] constructed a metric version of the Mahler measure. Let us briefly recall this construction. Write

$$(1.2) \quad \mathcal{X}(\overline{\mathbb{Q}}^\times) = \{(\alpha_1, \alpha_2, \dots) : \alpha_n = 1 \text{ for all but finitely many } n\}$$

to denote the restricted infinite direct product of $\overline{\mathbb{Q}}^\times$. Let $\tau : \mathcal{X}(\overline{\mathbb{Q}}^\times) \rightarrow \overline{\mathbb{Q}}^\times$ be defined by

$$\tau(\alpha_1, \alpha_2, \dots) = \prod_{n=1}^{\infty} \alpha_n$$

and note that τ is indeed a group homomorphism. The *metric Mahler measure* M_1 of α is given by

$$M_1(\alpha) = \inf \left\{ \sum_{n=1}^{\infty} M(\alpha_n) : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \right\}.$$

We note that the infimum in the definition of $M_1(\alpha)$ is taken over all ways of writing α as a product of elements in $\overline{\mathbb{Q}}^\times$. As a result of this construction, the function M_1 satisfies that triangle inequality

$$(1.3) \quad M_1(\alpha\beta) \leq M_1(\alpha) + M_1(\beta)$$

for all $\alpha, \beta \in \overline{\mathbb{Q}}^\times$. It can be shown that $M_1(\alpha) = 0$ if and only if α is a root of unity, and moreover, M_1 is well-defined on the quotient group $\mathcal{G} = \overline{\mathbb{Q}}^\times / \text{Tor}(\overline{\mathbb{Q}}^\times)$. Using (1.1) and (1.3), we find that the map $(\alpha, \beta) \mapsto M_1(\alpha\beta^{-1})$ is a metric on \mathcal{G} . It is noted in [3] that this map yields the discrete topology if and only if Lehmer's conjecture is true.

Following the strategy of [3], Fili and the author [4] examined a non-Archimedean version of the metric Mahler measure. That is, define the *ultrametric Mahler measure* M_∞ of α by

$$M_\infty(\alpha) = \inf \left\{ \max_{n \geq 1} M(\alpha_n) : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \right\},$$

replacing the sum in the definition of M_1 by a maximum. In this case, M_∞ has the strong triangle inequality

$$M_\infty(\alpha\beta) \leq \max\{M_\infty(\alpha), M_\infty(\beta)\}$$

for all $\alpha, \beta \in \overline{\mathbb{Q}}^\times$. Once again, we are able to verify that M_∞ is well-defined on \mathcal{G} . Here, the map $(\alpha, \beta) \mapsto M_\infty(\alpha\beta^{-1})$ yields a non-Archimedean metric on \mathcal{G} which induces the discrete topology if and only if Lehmer's conjecture is true.

In view of the definitions of M_1 and M_∞ , it is natural to define a collection of intermediate metric Mahler measures in the following way. If $x \in (0, \infty]$, we define $M_x : \mathcal{X}(\overline{\mathbb{Q}}^\times) \rightarrow [0, \infty)$ by

$$M_x(\alpha_1, \alpha_2, \dots) = \begin{cases} \left(\sum_{n=1}^{\infty} M(\alpha_n)^x \right)^{1/x} & \text{if } x \in (0, \infty) \\ \max_{n \geq 1} \{M(\alpha_n)\} & \text{if } x = \infty. \end{cases}$$

In the case that $x \geq 1$, we see that $M_x(\alpha_1, \alpha_2, \dots)$ is the L^x norm on the vector $(M(\alpha_1), M(\alpha_2), \dots)$. Then we define the x -metric Mahler measure by

$$(1.4) \quad M_x(\alpha) = \inf \{M_x(\bar{\alpha}) : \bar{\alpha} \in \tau^{-1}(\alpha)\}$$

and note that this definition generalizes those of M_1 and M_∞ . Indeed, the 1- and ∞ -metric Mahler measures are simply the metric and ultrametric Mahler measures, respectively.

In [3], Dubickas and Smyth showed that if Lehmer's conjecture is true, then the infimum in the definition of $M_1(\alpha)$ must always be achieved. The author [10] was able to verify that the infima in $M_1(\alpha)$ and $M_\infty(\alpha)$ are achieved even without the assumption of Lehmer's conjecture. Moreover, this infimum must always be attained in a relatively simple subgroup of $\overline{\mathbb{Q}}^\times$. In particular, if K is a number field we write

$$\text{Rad}(K) = \left\{ \alpha \in \overline{\mathbb{Q}}^\times : \alpha^r \in K \text{ for some } r \in \mathbb{N} \right\}.$$

For any algebraic number α , let K_α denote the Galois closure of $\mathbb{Q}(\alpha)$ over \mathbb{Q} . We showed in [10] that the infimum in both $M_1(\alpha)$ and $M_\infty(\alpha)$ is always attained by some

$$\bar{\alpha} \in \tau^{-1}(\alpha) \cap \mathcal{X}(\text{Rad}(K_\alpha)).$$

where $\mathcal{X}(\text{Rad}(K_\alpha))$ is defined similarly to $\mathcal{X}(\overline{\mathbb{Q}}^\times)$ in (1.2). Not surprisingly, the same argument can be used to establish the analog for all values of x .

Theorem 1.1. *Suppose α is a non-zero algebraic number and $x \in (0, \infty]$. Then there exists a point $\bar{\alpha} \in \tau^{-1}(\alpha) \cap \mathcal{X}(\text{Rad}(K_\alpha))$ such that $M_x(\alpha) = M_x(\bar{\alpha})$.*

We now turn our attention momentarily to the computation of some values of $M_x(\alpha)$. First define

$$C(\alpha) = \inf \{M(\gamma) : \gamma \in K_\alpha \setminus \text{Tor}(\overline{\mathbb{Q}}^\times)\}$$

and note that by Northcott's Theorem [9], the infimum on the right hand side of this definition is always achieved. In particular, this means that $C(\alpha) > 0$.

The author [11] gave a strategy for reducing the computation of $M_\infty(\alpha)$ to a finite set. The method uses the *modified Mahler measure*

$$(1.5) \quad \bar{M}(\alpha) = \inf \{M(\zeta\alpha) : \zeta \in \text{Tor}(\overline{\mathbb{Q}}^\times)\}$$

and gives the value of M_∞ in terms of \bar{M} . Although \bar{M} requires taking an infimum over an infinite set, it is often very reasonable to calculate. Indeed, the infimum on the right hand side of (1.5) is always attained at a root of unity ζ that makes $\deg(\zeta\alpha)$ as small as possible. This function \bar{M} arises again when computing $M_x(\alpha)$ for small x in a more straightforward way than in [11].

Theorem 1.2. *If α is a non-zero algebraic number and x is a positive real number satisfying*

$$(1.6) \quad x \cdot (\log \bar{M}(\alpha) - \log C(\alpha)) \leq \log 2$$

then $M_x(\alpha) = \bar{M}(\alpha)$.

As we will discuss in detail in section 2, the construction given by (1.4) is not unique to the Mahler measure. Suppose $\phi : \overline{\mathbb{Q}}^\times \rightarrow [0, \infty)$ satisfies

$$(1.7) \quad \phi(1) = 0 \quad \text{and} \quad \phi(\alpha) = \phi(\alpha^{-1}) \quad \text{for all } \alpha \in \overline{\mathbb{Q}}^\times,$$

and write

$$\phi_x(\alpha_1, \alpha_2, \dots) = \begin{cases} \left(\sum_{n=1}^{\infty} \phi(\alpha_n)^x \right)^{1/x} & \text{if } x \in (0, \infty) \\ \max_{n \geq 1} \{\phi(\alpha_n)\} & \text{if } x = \infty. \end{cases}$$

Generalizing the metric Mahler measure, let ϕ_x be defined by

$$(1.8) \quad \phi_x(\alpha) = \inf \{ \phi_x(\bar{\alpha}) : \bar{\alpha} \in \tau^{-1}(\alpha) \}.$$

We now write $\mathcal{S}(M)$ to denote the set of all functions ϕ satisfying (1.7) such that $\phi_x(\alpha) = M_x(\alpha)$ for all $\alpha \in \overline{\mathbb{Q}}^\times$ and $x \in (0, \infty]$. We are able to show that \bar{M} belongs to $\mathcal{S}(M)$. Moreover, it is a consequence of Theorem 1.2 that \bar{M} is the minimal element of $\mathcal{S}(M)$.

Corollary 1.3. *We have that $\bar{M} \in \mathcal{S}(M)$. Moreover, if $\psi \in \mathcal{S}(M)$ then $\psi(\alpha) \geq \bar{M}(\alpha)$ for all $\alpha \in \overline{\mathbb{Q}}^\times$.*

We now ask if the map $x \mapsto M_x(\alpha)$ is continuous on $\mathbb{R}_{>0}$ for every algebraic number α . We recall that Theorem 1.1 asserts that, for each x , there exists a point $\bar{\alpha} \in \tau^{-1}(\alpha)$ that attains the infimum in the definition of $M_x(\alpha)$. If the infimum is achieved at the same point $(\alpha_1, \alpha_2, \dots)$ for all real x , then we have that

$$M_x(\alpha) = \left(\sum_{n=1}^N M(\alpha_n)^x \right)^{1/x}$$

which clearly defines a continuous function. Unfortunately, using the example of $M_x(p^2)$ for a rational prime p , we see that this is not the case.

Theorem 1.4. *Let p be a rational prime and assume that $(\alpha_1, \alpha_2, \dots) \in \tau^{-1}(p^2)$ with $M_x(p^2) = M_x(\alpha_1, \alpha_2, \dots)$.*

- (i) *If $x \cdot (\log \log(p^2) - \log \log 2) < \log 2$ then precisely one point α_n differs from a root of unity.*
- (ii) *If $x > 1$ then at least two points α_n differ from a root of unity.*

Although the infimum in $M_x(\alpha)$ is not achieved at the same point for all x , we are able to prove that $x \mapsto M_x(\alpha)$ is continuous for all α .

Theorem 1.5. *If α is a non-zero algebraic number then the map $x \mapsto M_x(\alpha)$ is continuous on the positive real numbers.*

It is worth noting that continuity appears to be somewhat special to the Mahler measure. That is, we cannot expect an arbitrary function ϕ satisfying (1.7) to be such that $x \mapsto \phi_x(\alpha)$ is continuous. Even making a slight modification to the Mahler measure causes continuity to fail. For example, define the *Weil height* of $\alpha \in \overline{\mathbb{Q}}^\times$ by

$$h(\alpha) = \frac{M(\alpha)}{\deg \alpha}$$

and note that, in view of our remarks about the Mahler measure, $h(\alpha) = 0$ if and only if α is a root of unity. In fact, it is well-known that

$$(1.9) \quad h(\alpha) = h(\zeta\alpha)$$

for all roots of unity ζ . Moreover, we have that $h(\alpha) = h(\alpha^{-1})$ for all $\alpha \in \overline{\mathbb{Q}}^\times$ so that h satisfies (1.7). Unlike the Mahler measure, we know how to compute $h_x(\alpha)$ for every x and α .

Theorem 1.6. *If α is a non-zero algebraic number then*

$$h_x(\alpha) = \begin{cases} h(\alpha) & \text{if } x \leq 1 \\ 0 & \text{if } x > 1. \end{cases}$$

As we have noted, Theorem 1.6 does indeed show that $x \mapsto h_x(\alpha)$ is possibly discontinuous. More specifically, it is continuous if and only if α is a root of unity.

2. HEIGHTS ON ABELIAN GROUPS

In this section, we generalize our x -metric Mahler measure construction to a very broad class of functions on an abelian group G by exploring definition (1.8) in more detail. We are able to establish some basic properties in this situation that we can use to prove our main results.

Let G be a multiplicatively written abelian group. We say that $\phi : G \rightarrow [0, \infty)$ is a (*logarithmic*) *height* on G if

- (i) $\phi(1) = 0$, and
- (ii) $\phi(\alpha) = \phi(\alpha^{-1})$ for all $\alpha \in G$.

If ψ is another height on G , we follow the conventional notation that

$$\phi = \psi \quad \text{or} \quad \phi \leq \psi$$

when $\phi(\alpha) = \psi(\alpha)$ or $\phi(\alpha) \leq \psi(\alpha)$ for all $\alpha \in G$, respectively. We write

$$Z(\phi) = \{\alpha \in G : \phi(\alpha) = 0\}$$

to denote the *zero set* of ϕ .

If x is a positive real number then we say that ϕ has the *x -triangle inequality* if

$$\phi(\alpha\beta) \leq (\phi(\alpha)^x + \phi(\beta)^x)^{1/x}$$

for all $\alpha, \beta \in G$. We say that ϕ has the *∞ -triangle inequality* if

$$\phi(\alpha\beta) \leq \max\{\phi(\alpha), \phi(\beta)\}$$

for all $\alpha, \beta \in G$. For appropriate x , we say that these functions are *x -metric heights*. We observe that the 1-triangle inequality is simply the classical triangle inequality while the ∞ -triangle inequality is the strong triangle inequality. We also obtain the following ordering of the x -triangle inequalities.

Lemma 2.1. *Suppose that G is an abelian group and that $x, y \in (0, \infty]$ with $x \geq y$. If ϕ is an x -metric height on G then ϕ is also a y -metric height on G .*

Proof. If a, b and q are real numbers with $a, b \geq 0$ and $q \geq 1$, then it is easily verified that

$$(2.1) \quad a^q + b^q \leq (a + b)^q.$$

Let us now assume that ϕ has the x -triangle inequality and that $\alpha, \beta \in G$. If $x = y = \infty$ then the lemma is completely trivial. If $x = \infty$ and $y < \infty$ then we have that

$$\phi(\alpha\beta) \leq \max\{\phi(\alpha), \phi(\beta)\} = \max\{\phi(\alpha)^y, \phi(\beta)^y\}^{1/y} \leq (\phi(\alpha)^y + \phi(\beta)^y)^{1/y}$$

so that the result follows easily as well. Hence, we assume now that $\infty > x \geq y$. In this situation, we have that $x/y \geq 1$. Therefore, by (2.1) we have that

$$(\phi(\alpha)^y + \phi(\beta)^y)^{x/y} \geq \phi(\alpha)^x + \phi(\beta)^x$$

and it follows that

$$(\phi(\alpha)^y + \phi(\beta)^y)^{1/y} \geq (\phi(\alpha)^x + \phi(\beta)^x)^{1/x}.$$

Hence, we have that $\phi(\alpha\beta) \leq (\phi(\alpha)^y + \phi(\beta)^y)^{1/y}$ so that ϕ has the y -triangle inequality. \square

We now observe that each x -metric height is well-defined on the quotient group $G/Z(\phi)$. In the case that $x \geq 1$, the map $(\alpha, \beta) \mapsto \phi(\alpha\beta^{-1})$ defines a metric on $G/Z(\phi)$.

Theorem 2.2. *If $\phi : G \rightarrow [0, \infty)$ is an x -metric height for some $x \in (0, \infty]$ then*

- (i) $Z(\phi)$ is a subgroup of G .
- (ii) $\phi(\zeta\alpha) = \phi(\alpha)$ for all $\alpha \in G$ and $\zeta \in Z(\phi)$. That is, ϕ is well-defined on the quotient $G/Z(\phi)$.
- (iii) If $x \geq 1$, then the map $(\alpha, \beta) \mapsto \phi(\alpha\beta^{-1})$ defines a metric on $G/Z(\phi)$.

Proof. We first establish (i). Obviously, we have that $1 \in Z(G)$ by definition of height. Further, if $\phi(\alpha) = 0$ then again by definition of height we know that $\phi(\alpha^{-1}) = 0$. If $\alpha, \beta \in Z(G)$ then using the x triangle inequality we obtain

$$\phi(\alpha\beta) \leq (\phi(\alpha)^x + \phi(\beta)^x)^{1/x} = 0.$$

Therefore, $\alpha\beta \in Z(G)$ so that $Z(G)$ forms a subgroup.

To prove (ii), we see that the x -triangle inequality yields

$$\begin{aligned} \phi(\alpha) &= \phi(\zeta^{-1}\zeta\alpha) \\ &\leq (\phi(\zeta^{-1})^x + \phi(\zeta\alpha)^x)^{1/x} \\ &= \phi(\zeta\alpha) \\ &\leq (\phi(\zeta)^x + \phi(\alpha)^x)^{1/x} \\ &= \phi(\alpha) \end{aligned}$$

implying that $\phi(\alpha) = \phi(\zeta\alpha)$.

Finally, if $x \geq 1$ then Lemma 2.1 implies that ϕ has the triangle inequality. It then follows immediately that the map $(\alpha, \beta) \mapsto \phi(\alpha\beta^{-1})$ is a metric on $G/Z(\phi)$. \square

We are careful to note that if $x < 1$ then the map $(\alpha, \beta) \mapsto \phi(\alpha\beta^{-1})$ does not, in general, form a metric on $G/Z(\phi)$. In this case, the x -triangle inequality is indeed weaker than the triangle inequality, so we cannot expect the above map to form a metric except in trivial cases.

We now follow the method of Dubickas and Smyth for creating a metric from the Mahler measure. Write

$$\mathcal{X}(G) = \{(\alpha_1, \alpha_2, \dots) : \alpha_n = 1 \text{ for almost every } n\}$$

and, as before, let $\tau : \mathcal{X}(G) \rightarrow G$ be defined by

$$\tau(\alpha_1, \alpha_2, \dots) = \prod_{n=1}^{\infty} \alpha_n$$

so that τ is a group homomorphism. For each point $x \in (0, \infty]$ we define the map $\phi_x : \mathcal{X}(G) \rightarrow [0, \infty]$ by

$$\phi_x(\alpha_1, \alpha_2, \dots) = \begin{cases} \left(\sum_{n=1}^{\infty} \phi(\alpha_n)^x \right)^{1/x} & \text{if } x \in (0, \infty) \\ \max_{n \geq 1} \{\phi(\alpha_n)\} & \text{if } x = \infty. \end{cases}$$

Then we define the x -metric version of ϕ_x of ϕ by

$$\phi_x(\alpha) = \inf \{ \phi_x(\bar{\alpha}) : \bar{\alpha} \in \tau^{-1}(\alpha) \}.$$

It is immediately clear that if ψ is another height on G with $\phi \geq \psi$, then $\phi_x \geq \psi_x$ for all x . Among other things, we see that ϕ_x is indeed an x -metric height on G .

Theorem 2.3. *If $\phi : G \rightarrow [0, \infty)$ is a height on G and $x \in (0, \infty]$ then*

- (i) ϕ_x is an x -metric height on G with $\phi_x \leq \phi$.
- (ii) If ψ is an x -metric height with $\psi \leq \phi$ then $\psi \leq \phi_x$.
- (iii) $\phi = \phi_x$ if and only if ϕ is an x -metric height. In particular, $(\phi_x)_x = \phi_x$.
- (iv) If $y \in (0, x]$ then $\phi_y \geq \phi_x$.

Proof. For the proofs of (i)-(iii), we will assume that $x < \infty$. The proofs for the case $x = \infty$ are quite similar to the proofs for other cases so we will not include them here. See [4] for detailed proofs when $x = \infty$.

To prove (i), let $\alpha, \beta \in G$. We observe that if $(\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha)$ and $(\beta_1, \beta_2, \dots) \in \tau^{-1}(\beta)$ then it is obvious that

$$\alpha\beta = \left(\prod_{n=1}^{\infty} \alpha_n \right) \left(\prod_{n=1}^{\infty} \beta_n \right).$$

We may also write

$$\alpha\beta = \prod_{n=1}^{\infty} \alpha_n \beta_n$$

implying that $\tau(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots) = \alpha\beta$. In other words, we have that

$$(2.2) \quad (\alpha_1, \beta_1, \alpha_2, \beta_2, \dots) \in \tau^{-1}(\alpha\beta).$$

This yields that

$$\begin{aligned}
 \phi_x(\alpha\beta)^x &= \inf\{\phi_x(\gamma_1, \gamma_2, \dots)^x : (\gamma_1, \gamma_2, \dots) \in \tau^{-1}(\alpha\beta)\} \\
 &= \inf\{\phi_x(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots)^x : \alpha_n, \beta_n \in G, (\alpha_1, \beta_1, \dots) \in \tau^{-1}(\alpha\beta)\} \\
 (2.3) \quad &\leq \inf\{\phi_x(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots)^x : (\alpha_1, \dots) \in \tau^{-1}(\alpha), (\beta_1, \dots) \in \tau^{-1}(\beta)\}.
 \end{aligned}$$

We note that

$$\begin{aligned}
 \phi_x(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots)^x &= \sum_{n=1}^{\infty} (\phi(\alpha_n)^x + \phi(\beta_n)^x) \\
 &= \sum_{n=1}^{\infty} \phi(\alpha_n)^x + \sum_{n=1}^{\infty} \phi(\beta_n)^x \\
 &= \phi_x(\alpha_1, \dots)^x + \phi_x(\beta_1, \dots)^x.
 \end{aligned}$$

Then using (2.3) we find that

$$\begin{aligned}
 \phi(\alpha\beta)^x &\leq \inf\{\phi_x(\alpha_1, \dots)^x + \phi_x(\beta_1, \dots)^x : (\alpha_1, \dots) \in \tau^{-1}(\alpha), (\beta_1, \dots) \in \tau^{-1}(\beta)\} \\
 &= \inf\{\phi_x(\alpha_1, \dots)^x : (\alpha_1, \dots) \in \tau^{-1}(\alpha)\} \\
 &\quad + \inf\{\phi_x(\beta_1, \dots)^x : (\beta_1, \dots) \in \tau^{-1}(\beta)\} \\
 &= \phi_x(\alpha)^x + \phi_x(\beta)^x
 \end{aligned}$$

and it follows that

$$\phi_x(\alpha\beta) \leq (\phi_x(\alpha)^x + \phi_x(\beta)^x)^{1/x}.$$

To complete the proof of (i), we observe that $(\alpha, 1, 1, \dots) \in \tau^{-1}(\alpha)$ so that $\phi_x(\alpha) \leq \phi(\alpha)$ for all $\alpha \in G$.

To prove (ii), we note that

$$\begin{aligned}
 \phi_x(\alpha) &= \inf \left\{ \left(\sum_{n=1}^N \phi(\alpha_n)^x \right)^{1/x} : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \right\} \\
 &\geq \inf \left\{ \left(\sum_{n=1}^N \psi(\alpha_n)^x \right)^{1/x} : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \right\} \\
 &\geq \psi(\alpha)
 \end{aligned}$$

where the last inequality follows from the fact that ψ has the x -triangle inequality.

To prove (iii), we first observe that if $\phi = \phi_x$ then clearly ϕ is an x -metric height. If ϕ is already a metric height, then by (ii), we obtain that $\phi \leq \phi_x$. But we always have $\phi_x \leq \phi$ so the result follows. Of course, ϕ_x is an x -metric height so this yields immediately $\phi_x = (\phi_x)_x$.

To establish (iv), we see that

$$\begin{aligned}
 \phi_y(\alpha) &= \inf \left\{ \left(\sum_{n=1}^N \phi(\alpha_n)^y \right)^{1/y} : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \right\} \\
 &= \inf \left\{ \left(\sum_{n=1}^N \phi(\alpha_n)^y \right)^{\frac{x}{y} \cdot \frac{1}{x}} : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \right\}.
 \end{aligned}$$

But we have that $x \geq y$ so that $x/y \geq 1$. Therefore, by Lemma 2.1 we have that

$$\left(\sum_{n=1}^N \phi(\alpha_n)^y \right)^{x/y} \geq \sum_{n=1}^N \phi(\alpha_n)^x$$

which yields $\phi_y(\alpha) \geq \phi_x(\alpha)$. \square

For a given height ϕ on G , let $\mathcal{S}(\phi)$ denote the set of all heights ψ on G such that $\psi_x = \phi_x$ for all $x \in (0, \infty]$. Further, define the height ϕ_0 by

$$(2.4) \quad \phi_0(\alpha) = \lim_{x \rightarrow 0^+} \phi_x(\alpha).$$

By (i) of Theorem 2.3, we know that $\phi_x \leq \phi$ for all x . Moreover, (iv) of the same theorem states that $x \mapsto \phi_x(\alpha)$ is non-increasing. This means that the limit on the right hand side of (2.4) does indeed exist and

$$(2.5) \quad \phi_0 \geq \phi_x$$

for all $x \in (0, \infty]$. We now observe that ϕ_0 is the minimal element of $\mathcal{S}(\phi)$.

Theorem 2.4. *If ϕ is a height on G then $\phi_0 \in \mathcal{S}(\phi)$. Moreover, if $\psi \in \mathcal{S}(\phi)$ then $\psi \geq \phi_0$.*

Proof. As we have noted, $\phi_0 \geq \phi_x$ for all x . Hence, we obtain immediately that $(\phi_0)_x \geq (\phi_x)_x = \phi_x$. On the other hand, we know that $\phi_x \leq \phi$ so that

$$\phi_0(\alpha) = \lim_{x \rightarrow 0^+} \phi_x(\alpha) \leq \phi(\alpha)$$

for all $\alpha \in G$. In other words, we have that $\phi_0 \leq \phi$ so that $(\phi_0)_x \leq \phi_x$ establishing the first statement of the theorem.

To prove the second statement, assume that $\psi \in \mathcal{S}(\phi)$ so that $\phi_x = \psi_x$ for all x . Hence we have that

$$\phi_0(\alpha) = \lim_{x \rightarrow 0^+} \phi_x(\alpha) = \lim_{x \rightarrow 0^+} \psi_x(\alpha) \leq \psi(\alpha)$$

for all $\alpha \in G$ verifying the theorem. \square

We now define the *modified version* of ϕ by

$$\bar{\phi}(\alpha) = \inf\{\phi(\zeta\alpha) : \zeta \in Z(\phi)\}.$$

In the case of the Mahler measure, we have stated in the introduction that $\bar{\phi} = \phi_0$. However, in the general case, we can conclude only that $\bar{\phi}$ belongs to $\mathcal{S}(\phi)$.

Theorem 2.5. *If ϕ is a height on G then $\bar{\phi} \in \mathcal{S}(\phi)$.*

Proof. We must show that $\bar{\phi}_x = \phi_x$ for all $x \in (0, \infty]$. Since $1 \in Z(\phi)$, we have immediately that $\bar{\phi} \leq \phi$, which means that

$$\bar{\phi}_x \leq \phi_x.$$

Now for any $\alpha \in G$, we have that

$$\phi_x(\alpha) \leq \inf\{(\phi(\zeta^{-1})^x + \phi(\zeta\alpha)^x)^{1/x} : \zeta \in Z(\phi)\} = \inf\{\phi(\zeta\alpha) : \zeta \in Z(\phi)\} = \bar{\phi}(\alpha)$$

implying that $\phi_x \leq \bar{\phi}$. Then taking x -metric versions and using (iii) of Theorem 2.3 we find that

$$\phi_x = (\phi_x)_x \leq \bar{\phi}_x$$

completing the proof. \square

We may now ask what we can say about the map $x \mapsto \phi_x(\alpha)$ for fixed ϕ and α . As we have noted, this map is non-increasing for all α . Since $\phi_x(\alpha)$ is bounded from above and below by constants not depending on x , both left and right hand limits exist at every point. Moreover, we always have

$$\lim_{x \rightarrow \bar{x}^-} \phi_x(\alpha) \geq \phi_{\bar{x}}(\alpha) \geq \lim_{x \rightarrow \bar{x}^+} \phi_x(\alpha)$$

when $\bar{x} > 0$. We say that a map $f : \mathbb{R} \rightarrow \mathbb{R}$ is *left* or *right semi-continuous* at a point $\bar{x} \in \mathbb{R}$ if

$$\lim_{x \rightarrow \bar{x}^-} f(x) = f(\bar{x}) \quad \text{or} \quad \lim_{x \rightarrow \bar{x}^+} f(x) = f(\bar{x}),$$

respectively. Indeed, f is continuous at \bar{x} if and only if f is both left and right semi-continuous at \bar{x} . Although it is a consequence of Theorem 1.6 that $x \mapsto \phi_x(\alpha)$ is not continuous in general, we can prove the following partial result.

Theorem 2.6. *If ϕ is a height on G and $\alpha \in G$, then the map $x \mapsto \phi_x(\alpha)$ is left semi-continuous on the positive real numbers.*

Proof. We already know that $\lim_{x \rightarrow \bar{x}^-} \phi_x(\alpha) \geq \phi_{\bar{x}}(\alpha)$ so we assume that

$$\lim_{x \rightarrow \bar{x}^-} \phi_x(\alpha) > \phi_{\bar{x}}(\alpha).$$

Therefore, there exists $\varepsilon > 0$ such that

$$(2.6) \quad \lim_{x \rightarrow \bar{x}^-} \phi_x(\alpha) > \phi_{\bar{x}}(\alpha) + \varepsilon.$$

By definition of $\phi_{\bar{x}}$, we may choose points $\alpha_1, \dots, \alpha_N \in G$ such that $\alpha = \alpha_1 \cdots \alpha_N$ and

$$\phi_{\bar{x}}(\alpha) + \varepsilon \geq \left(\sum_{n=1}^N \phi(\alpha_n)^{\bar{x}} \right)^{1/\bar{x}},$$

and define the function f_ε by

$$f_\varepsilon(x) = \left(\sum_{n=1}^N \phi(\alpha_n)^x \right)^{1/x}.$$

This yields

$$(2.7) \quad f_\varepsilon(\bar{x}) \leq \phi_{\bar{x}}(\alpha) + \varepsilon \quad \text{and} \quad f_\varepsilon(x) \geq \phi_x(\alpha) \text{ for all } x.$$

Also, since f_ε is continuous, we have that

$$(2.8) \quad f_\varepsilon(\bar{x}) = \lim_{x \rightarrow \bar{x}^-} f_\varepsilon(x).$$

Combining (2.6), (2.7) and (2.8) we obtain that

$$f_\varepsilon(\bar{x}) = \lim_{x \rightarrow \bar{x}^-} f_\varepsilon(x) \geq \lim_{x \rightarrow \bar{x}^-} \phi_x(\alpha) > \phi_{\bar{x}}(\alpha) + \varepsilon \geq f_\varepsilon(\bar{x})$$

which is a contradiction. □

3. THE INIFIMUM IN $M_x(\alpha)$

Our proof of Theorem 1.1 will require the use of two results from [10]. The first of these is Theorem 2.1 of [10], which shows that for any point $\bar{\alpha} \in \tau^{-1}(\alpha)$, there exists another point $\bar{\beta} \in \tau^{-1}(\alpha) \cup \mathcal{X}(\text{Rad}(K_\alpha))$ which has pointwise smaller Mahler measures. We state the Theorem using the notation of [10].

Theorem 3.1. *If $\alpha, \alpha_1, \dots, \alpha_N$ are non-zero algebraic numbers with $\alpha = \alpha_1 \cdots \alpha_N$ then there exists a root of unity ζ and algebraic numbers β_1, \dots, β_N satifying*

- (i) $\alpha = \zeta \beta_1 \cdots \beta_N$,
- (ii) $\beta_n \in \text{Rad}(K_\alpha)$ for all n ,
- (iii) $M(\beta_n) \leq M(\alpha_n)$ for all n .

In view of Theorem 3.1, for each x , we need only consider only points $\bar{\alpha} \in \tau^{-1}(\alpha) \cup \mathcal{X}(\text{Rad}(K_\alpha))$ in the definition of $M_x(\alpha)$. In other words, in the case of $x < \infty$, the definition of $M_x(\alpha)$ may be rewritten

$$(3.1) \quad M_x(\alpha) = \inf \left\{ \left(\sum_{n=1}^{\infty} M(\alpha_n)^x \right)^{1/x} : (\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha) \cup \mathcal{X}(\text{Rad}(K_\alpha)) \right\}.$$

Similar remarks apply in the case that $x = \infty$. Therefore, it will be useful to have some control of the Mahler measures in the subgroup $\text{Rad}(K_\alpha)$. For this purpose, we borrow Lemma 3.1 of [10].

Lemma 3.2. *Let K be a Galois extension of \mathbb{Q} . If $\gamma \in \text{Rad}(K)$ then there exists a root of unity ζ and $L, S \in \mathbb{N}$ such that $\zeta \gamma^L \in K$ and*

$$M(\gamma) = M(\zeta \gamma^L)^S.$$

In particular, the set

$$\{M(\gamma) : \gamma \in \text{Rad}(K), M(\gamma) \leq B\}$$

is finite for every $B \geq 0$.

It is an easy consequence of Lemma 3.2 that $M(\gamma)$ is bounded below by the Mahler measure of an element in K . Indeed, we have that

$$M(\gamma) = M(\zeta \gamma^L)^S \geq M(\zeta \gamma^L)$$

and $\zeta \gamma^L \in K$. In particular, we recall that $C(\alpha)$ denotes the minimum Mahler measure in the field K_α . We now see easily that

$$(3.2) \quad M(\gamma) \geq C(\alpha)$$

for all $\gamma \in \text{Rad}(K_\alpha) \setminus \text{Tor}(\overline{\mathbb{Q}}^\times)$. We are now prepared to prove Theorem 1.1.

Proof of Theorem 1.1. By the results of [10], we know that the theorem holds for $x = \infty$, so we may assume that $x < \infty$. Further, select a real number $B > M_x(\alpha)$. In view of Theorem 3.1, we know that $M_x(\alpha)$ is the infimum of

$$(3.3) \quad \left(\sum_{n=1}^N M(\alpha_n)^x \right)^{1/x}$$

over the set of all $N \in \mathbb{N}$ and all points $\alpha_1, \dots, \alpha_N \in \overline{\mathbb{Q}}^\times$ such that

- (i) $\alpha = \alpha_1 \cdots \alpha_N$,
- (ii) At most one point α_n is a root of unity,

(iii) $\alpha_n \in \text{Rad}(K_\alpha)$ for all n , and

(iv) $\left(\sum_{n=1}^N M(\alpha_n)^x\right)^{1/x} \leq B$.

We will show that the set of all values of (3.3) is finite for $\alpha_1, \dots, \alpha_N$ satisfying conditions (i)-(iv).

We must first give an upper bound on N . We know that at least $N - 1$ of the points $\alpha_1, \dots, \alpha_N$ are not roots of unity. For all such points, we have that

$$M(\alpha_n) \geq C(\alpha)$$

by (3.2). Combining this with (iv), we obtain that

$$B \geq \left(\sum_{n=1}^N M(\alpha_n)^x\right)^{1/x} \geq (N-1)^{1/x} C(\alpha)$$

which yields

$$(3.4) \quad N \leq 1 + \left(\frac{B}{C(\alpha)}\right)^x.$$

Also by (iv), it follows that $M(\alpha_n) \leq B$ for all n . Moreover, since $\alpha_n \in \text{Rad}(K_\alpha)$, the second statement of Lemma 3.2 implies that there are only finitely many possible values for $M(\alpha_n)$ for each n . Since N is bounded above by the right hand side of (3.4), it follows that there are only finitely many possible values for (3.3) with $\alpha_1, \dots, \alpha_N$ satisfying (i)-(iv). We now know that $M_x(\alpha)$ is an infimum over a finite set, so the infimum must be achieved. \square

4. MINIMALITY OF \bar{M}

We first give the proof of Theorem 1.2 showing that $M_x(\alpha) = \bar{M}(\alpha)$ for sufficiently small values of x .

Proof of Theorem 1.2. By Theorem 2.5, we have immediately that $M_x(\alpha) = \bar{M}_x(\alpha)$ for all x , so it follows that

$$(4.1) \quad M_x(\alpha) \leq \bar{M}(\alpha).$$

Now we must prove the opposite inequality.

We know by Theorem 1.1 that there exist points $\alpha_1, \dots, \alpha_N \in \text{Rad}(K_\alpha)$ such that

$$\alpha = \alpha_1 \cdots \alpha_N \quad \text{and} \quad M_x(\alpha) = \left(\sum_{n=1}^N M(\alpha_n)^x\right)^{1/x}.$$

We know that α is not a root of unity, so at least one of $\alpha_1, \dots, \alpha_N$ is not a root of unity.

We now consider two cases. First, assume that precisely one of $\alpha_1, \dots, \alpha_N$ is not a root of unity. In other words, there exists a root of unity ζ and a point $\beta \in \text{Rad}(K_\alpha) \setminus \text{Tor}(\overline{\mathbb{Q}}^\times)$ such that $\alpha = \zeta\beta$ and

$$M_x(\alpha) = M(\beta).$$

Of course, we also have $\beta = \alpha\zeta^{-1}$ so that

$$\bar{M}(\alpha) \leq M(\alpha\zeta^{-1}) = M(\beta) = M_x(\alpha).$$

Combining this inequality with (4.1), the result follows.

Next, assume that at least two of $\alpha_1, \dots, \alpha_N$ are not a roots of unity. By Lemma 3.2, we know that $M(\alpha_n) \geq C(\alpha)$ whenever α_n is not a root of unity. Hence, we obtain that

$$M_x(\alpha) = \left(\sum_{n=1}^N M(\alpha_n)^x \right)^{1/x} \geq (2C(\alpha)^x)^{1/x}$$

so that

$$(4.2) \quad M_x(\alpha) \geq 2^{1/x} C(\alpha).$$

By our assumption, we have that

$$\frac{1}{x} \geq \frac{\log \bar{M}(\alpha) - \log C(\alpha)}{\log 2}$$

which implies that

$$\begin{aligned} 2^{1/x} &\geq 2^{\frac{\log \bar{M}(\alpha) - \log C(\alpha)}{\log 2}} \\ &= \exp(\log \bar{M}(\alpha) - \log C(\alpha)) \\ &= \frac{\exp(\log \bar{M}(\alpha))}{\exp(\log C(\alpha))} \\ &= \frac{\bar{M}(\alpha)}{C(\alpha)}. \end{aligned}$$

It now follows from (4.2) that

$$M_x(\alpha) \geq \bar{M}(\alpha)$$

completing the proof. \square

Next, we establish Corollary 1.3 showing that \bar{M} is minimal in the set $\mathcal{S}(M)$.

Proof of Corollary 1.3. We observe again by Theorem 2.5 that $\bar{M} \in \mathcal{S}(M)$. By Theorem 1.2, for all sufficiently small x , we have that $\bar{M}(\alpha) = M_x(\alpha)$. Hence, it follows that that

$$\bar{M}(\alpha) = \lim_{x \rightarrow 0^+} M_x(\alpha) = M_0(\alpha)$$

and the result follows from Theorem 2.4. \square

We begin our proof of Theorem 1.4 by giving a slight modification to Theorem 1.2. More specifically, it will be useful to consider what happens when the supposed inequality (1.6) is replaced by a strict inequality.

Lemma 4.1. *Let α be a non-zero algebraic number different from a root of unity and x a positive real number satisfying*

$$x \cdot (\log \bar{M}(\alpha) - \log C(\alpha)) < \log 2.$$

Then any point $(\alpha_1, \alpha_2, \dots) \in \tau^{-1}(\alpha)$ that achieves the infimum in the definition of $M_x(\alpha)$ has precisely one component α_n that is not a root of unity.

Proof. We recall first that

$$(4.3) \quad M_x(\alpha) \leq \bar{M}(\alpha)$$

by Theorem 2.5. Next, we note that

$$(4.4) \quad \frac{1}{x} > \frac{\log \bar{M}(\alpha) - \log C(\alpha)}{\log 2}.$$

Assume that $\alpha_1, \dots, \alpha_N \in \overline{\mathbb{Q}}^\times$ are such that

$$(4.5) \quad \alpha = \alpha_1 \cdots \alpha_N \quad \text{and} \quad M_x(\alpha) = \left(\sum_{n=1}^N M(\alpha_n)^x \right)^{1/x}.$$

and at least two of the points $\alpha_1, \dots, \alpha_N$ are not roots of unity. By Theorem 3.1, there exists a root of unity ζ and points $\beta_1, \dots, \beta_N \in \text{Rad}(K_\alpha)$ such that

$$\alpha = \zeta \beta_1 \cdots \beta_N \quad \text{and} \quad M(\beta_n) \leq M(\alpha_n)$$

for all n . If for any n we have that $M(\beta_n) < M(\alpha_n)$, then

$$M_x(\alpha) \leq \left(\sum_{n=1}^N M(\beta_n)^x \right)^{1/x} < \left(\sum_{n=1}^N M(\alpha_n)^x \right)^{1/x}$$

which contradicts the right hand side of (4.5). Therefore, we have that $M(\beta_n) = M(\alpha_n)$ for all n . In particular, at least two of the points β_1, \dots, β_N are not roots of unity. Furthermore, since each $\beta_n \in \text{Rad}(K_\alpha)$, we may apply Lemma 3.2 to see that $M(\beta_n) \geq C(\alpha)$ whenever β_n is not a root of unity. This yields

$$M_x(\alpha) = \left(\sum_{n=1}^N M(\beta_n)^x \right)^{1/x} \geq (2C(\alpha)^x)^{1/x}.$$

which implies that

$$M_x(\alpha) \geq 2^{1/x} C(\alpha).$$

However, we now have the strict inequality (4.4) which gives $2^{1/x} > \bar{M}(\alpha)/C(\alpha)$ and

$$M_x(\alpha) > \bar{M}(\alpha)$$

contradicting (4.3). Therefore, exactly one point among $\alpha_1, \dots, \alpha_N$ is not a root of unity. \square

Before we prove Theorem 1.4, we recall our remark that $\bar{M}(\alpha)$ is often very reasonable to compute so that Theorem 1.2 and Lemma 4.1 are useful in applications. The following proof is a typical example.

Proof of Theorem 1.4. Let $\alpha = p^2$. In order to prove (i), we wish to apply Lemma 4.1, so we must compute the values of $\bar{M}(\alpha)$ and $C(\alpha)$. We begin by observing that

$$\bar{M}(\alpha) = \inf\{M(\zeta\alpha) : \zeta \in \text{Tor}(\overline{\mathbb{Q}}^\times)\} = \inf\{\deg(\zeta\alpha) \cdot h(\zeta\alpha) : \zeta \in \text{Tor}(\overline{\mathbb{Q}}^\times)\}.$$

Then by (1.9), we obtain that

$$(4.6) \quad \bar{M}(\alpha) = h(\alpha) \cdot \inf\{\deg(\zeta\alpha) : \zeta \in \text{Tor}(\overline{\mathbb{Q}}^\times)\}.$$

It is clear that the infimum on the right hand side of (4.6) is achieved since it is an infimum over positive integers. More specifically, it is achieved by a root of unity ζ that makes $\deg(\zeta\alpha)$ as small as possible. In our case, α is rational, so this occurs when $\zeta = 1$ leaving

$$(4.7) \quad \bar{M}(\alpha) = \bar{M}(p^2) = M(p^2) = \log(p^2).$$

In addition, we know that $K_\alpha = \mathbb{Q}$ so that $C(\alpha) = \log 2$ which now gives

$$x \cdot (\log \bar{M}(\alpha) - \log C(\alpha)) = x \cdot (\log \log(p^2) - \log \log 2) < \log 2.$$

By Lemma 4.1, we know that any point $(\alpha_1, \alpha_2, \dots)$ that attains the infimum in $M_x(\alpha) = M_x(p^2)$ must have precisely one point α_n that is not a root of unity. This completes the proof of (i).

To prove (ii), we take $x > 1$ and assume that $(\alpha_1, \alpha_2, \dots)$ attains the infimum in the definition of $M_x(p^2)$ where at most one point α_n is different from a root of unity. Therefore, there exists a root of unity ζ and an algebraic number β such that

$$p^2 = \zeta\beta \quad \text{and} \quad M_x(p^2) = M(\beta).$$

Hence we find immediately that

$$M(\beta) = M_x(p^2) \leq (M(p)^x + M(p)^x)^{1/x} = 2^{1/x} \log p.$$

Since $x > 1$, this yields that

$$M(\beta) < 2 \log p.$$

On the other hand, we have that $\beta = \zeta^{-1}p^2$ so that, using (4.7), we obtain

$$M(\beta) = M(\zeta^{-1}p^2) \geq \bar{M}(p^2) = 2 \log p$$

which is a contradiction. Thus, at least two points among $(\alpha_1, \alpha_2, \dots)$ must not be roots of unity. \square

5. CONTINUITY OF $x \mapsto M_x(\alpha)$

We have already proved that, for any height function ϕ , the map $x \mapsto \phi_x(\alpha)$ is left semi-continuous. In general, we know that such functions are not always right semi-continuous. However, we are able to use Theorem 1.1 and our observations about the Mahler measure to establish right semi-continuity in this case.

Proof of Theorem 1.5. If α is a root of unity, then $M_x(\alpha) = 0$ for all x , so we may assume that α is not a root of unity. Furthermore, we know by Theorem 2.6 that this map is left semi-continuous at all points, so it remains only to show that it is right semi-continuous.

Now let $\bar{x} > 0$ be a real number, so we must show that

$$(5.1) \quad \lim_{y \rightarrow \bar{x}^+} M_y(\alpha) = M_{\bar{x}}(\alpha).$$

Since $x \mapsto M_x(\alpha)$ is decreasing, we know that the left hand side of (5.1) exists. Moreover, we have that

$$(5.2) \quad \lim_{y \rightarrow \bar{x}^+} M_y(\alpha) \leq M_{\bar{x}}(\alpha).$$

Now we select a point $y \in (\bar{x}, \bar{x} + 1]$. By Theorem 1.1, there must exist points

$$\alpha_1, \dots, \alpha_N \in \text{Rad}(K_\alpha) \setminus \text{Tor}(\overline{\mathbb{Q}}^\times)$$

and $\zeta \in \text{Tor}(\overline{\mathbb{Q}}^\times)$ such that

$$\alpha = \zeta \alpha_1 \cdots \alpha_N \quad \text{and} \quad M_y(\alpha) = \left(\sum_{n=1}^N M(\alpha_n)^y \right)^{1/y}.$$

Since $M_y(\alpha) \leq M(\alpha)$, we may assume without loss of generality that $M(\alpha_n) \leq M(\alpha)$ for all n . Furthermore, since α is not a root of unity, we know that $N \geq 1$.

For simplicity, we write now $a_n = M(\alpha_n)$ so that

$$M_y(\alpha) = \left(\sum_{n=1}^N a_n^y \right)^{1/y},$$

and note that by Lemma 3.2, we have that

$$(5.3) \quad a_n \geq C(\alpha) \text{ for all } n.$$

Next, we define the function f_y by

$$f_y(x) = \left(\sum_{n=1}^N a_n^x \right)^{1/x}$$

and note that f_y does indeed depend on y because the points ζ and $\alpha_1, \dots, \alpha_N$ depend on y . We now have immediately that

$$(5.4) \quad f_y(y) = M_y(\alpha).$$

Since $\alpha = \zeta \alpha_1 \cdots \alpha_N$, we know that

$$M_{\bar{x}}(\alpha) \leq \left(\sum_{n=1}^N M(\alpha_n)^{\bar{x}} \right)^{1/\bar{x}} = \left(\sum_{n=1}^N a_n^{\bar{x}} \right)^{1/\bar{x}} = f_y(\bar{x}),$$

and therefore, we obtain that

$$(5.5) \quad M_{\bar{x}}(\alpha) \leq f_y(\bar{x}).$$

We know that $a_n > 0$ for all n implying that $f_y(x) > 0$ for all x , so we may define the function $g_y(x) = \log f_y(x)$. Since f_y is differentiable on the positive real numbers, we know that g_y is as well. Therefore, we may apply the Mean Value Theorem to it on $[\bar{x}, y]$. Hence, there exists a point $c \in [\bar{x}, y]$ such that

$$g'_y(c) = \frac{g_y(y) - g_y(\bar{x})}{y - \bar{x}} = \frac{\log f_y(y) - \log f_y(\bar{x})}{y - \bar{x}}$$

and it follows from (5.4) and (5.5) that

$$(5.6) \quad g'_y(c) \leq \frac{\log M_y(\alpha) - \log M_{\bar{x}}(\alpha)}{y - \bar{x}}.$$

We now wish to take limits of both sides of (5.6) as y tends to \bar{x} from the right. However, it is possible that the limit of the left hand side either equals $-\infty$ or does not exist as $y \rightarrow \bar{x}^+$. To solve this problem, we wish to give a lower bound on $g'_y(c)$ that does not depend on y .

For any $x > 0$, we note that

$$\begin{aligned} g'_y(x) &= \frac{d}{dx} \log f_y(x) \\ &= \frac{d}{dx} \frac{1}{x} \left(\log \sum_{n=1}^N a_n^x \right) \\ &= \frac{1}{x^2} \left(x \cdot \frac{\left(\sum_{n=1}^N a_n^x \log a_n \right)}{\left(\sum_{n=1}^N a_n^x \right)} - \log \sum_{n=1}^N a_n^x \right). \end{aligned}$$

Then using (5.3), we have that

$$(5.7) \quad g'_y(x) \geq \frac{1}{x^2} \left(x \cdot \log C(\alpha) - \log \sum_{n=1}^N a_n^x \right).$$

Now we need to give an upper bound on $\sum_{n=1}^N a_n^x$. Recall that we must have $a_n = M(\alpha_n) \leq M(\alpha)$ for all n . Therefore, we have that

$$\sum_{n=1}^N a_n^x \leq NM(\alpha)^x.$$

But using (5.3) again, we find that

$$M(\alpha) \geq M_y(\alpha) = \left(\sum_{n=1}^N a_n^y \right)^{1/y} \geq (NC(\alpha)^y)^{1/y} = N^{1/y} C(\alpha).$$

We also know $C(\alpha) > 0$ and $y \in (\bar{x}, \bar{x} + 1]$ so that

$$N \leq \left(\frac{M(\alpha)}{C(\alpha)} \right)^y \leq \left(\frac{M(\alpha)}{C(\alpha)} \right)^{\bar{x}+1},$$

and therefore

$$\sum_{n=1}^N a_n^x \leq \frac{M(\alpha)^{x+\bar{x}+1}}{C(\alpha)^{\bar{x}+1}}.$$

It now follows that

$$-\log \sum_{n=1}^N a_n^x \geq -\log \left(\frac{M(\alpha)^{x+\bar{x}+1}}{C(\alpha)^{\bar{x}+1}} \right).$$

Combining this with (5.7), we obtain that

$$g'_y(x) \geq \frac{1}{x^2} \left(x \cdot \log C(\alpha) - \log \left(\frac{M(\alpha)^{x+\bar{x}+1}}{C(\alpha)^{\bar{x}+1}} \right) \right),$$

so we have shown that

$$(5.8) \quad g'_y(x) \geq \frac{x + \bar{x} + 1}{x^2} \log \left(\frac{C(\alpha)}{M(\alpha)} \right).$$

For simplicity, we now write $D(\alpha, \bar{x}, x)$ to denote the right hand side of (5.8). As a function of x , it is obvious that $D(\alpha, \bar{x}, x)$ is continuous for all $x > 0$. Hence, we may define

$$\mathcal{D}(\alpha, \bar{x}) = \min\{D(\alpha, \bar{x}, x) : x \in [\bar{x}, \bar{x} + 1]\}.$$

Now $\mathcal{D}(\alpha, \bar{x})$ is the desired lower bound on $g'_y(c)$ not depending on y .

Since $c \in [\bar{x}, y] \subset [\bar{x}, \bar{x} + 1]$, we may apply (5.6) and (5.8) to see that

$$\mathcal{D}(\alpha, \bar{x}) \leq D(\alpha, \bar{x}, c) \leq g'_y(c) \leq \frac{\log M_y(\alpha) - \log M_{\bar{x}}(\alpha)}{y - \bar{x}}.$$

By multiplying through by $y - \bar{x}$, we find that

$$(5.9) \quad (y - \bar{x})\mathcal{D}(\alpha, \bar{x}) \leq \log M_y(\alpha) - \log M_{\bar{x}}(\alpha)$$

holds for all $y \in (\bar{x}, \bar{x} + 1]$.

As we have noted, $\lim_{y \rightarrow \bar{x}^+} M_y(\alpha)$ exists. Since we have assumed that α is not a root of unity, we conclude from Theorem 1.1 that $M_y(\alpha) > 0$ for all y . It now follows that $\lim_{y \rightarrow \bar{x}^+} \log M_y(\alpha)$ also exists. Moreover, the term $\mathcal{D}(\alpha, \bar{x})$ is a real

number not depending on y , so the left hand side of (5.9) tends to zero as y tends to \bar{x} from the right. This leaves

$$\begin{aligned}
0 &= \lim_{y \rightarrow \bar{x}^+} ((y - \bar{x})\mathcal{D}(\alpha, \bar{x})) \\
&\leq \lim_{y \rightarrow \bar{x}^+} (\log M_y(\alpha) - M_{\bar{x}}(\alpha)) \\
&= \lim_{y \rightarrow \bar{x}^+} \log M_y(\alpha) - \lim_{y \rightarrow \bar{x}^+} \log M_{\bar{x}}(\alpha) \\
&= \lim_{y \rightarrow \bar{x}^+} \log M_y(\alpha) - \log M_{\bar{x}}(\alpha),
\end{aligned}$$

which yeilds

$$\log M_{\bar{x}}(\alpha) \leq \lim_{y \rightarrow \bar{x}^+} \log M_y(\alpha)$$

so that $M_{\bar{x}}(\alpha) \leq \lim_{y \rightarrow \bar{x}^+} M_y(\alpha)$ and the result follows by combining this with (5.2). \square

6. WEIL HEIGHT

Before we begin our proof of Theorem 1.6, we recall that if N is any integer, then it is well-known that

$$(6.1) \quad h(\alpha^N) = |N| \cdot h(\alpha)$$

for all algebraic numbers α . Using this fact, we are able to proceed with our proof.

Proof of Theorem 1.6. First assume that $x \leq 1$. By (i) of Theorem 2.3, we have that $h_x(\alpha) \leq h(\alpha)$. But also, it is well-known that h is already a 1-metric height. Therefore, (iii) of Theorem 2.3 implies that $h_1(\alpha) = h(\alpha)$. Then by (iv) of Theorem 2.3, we conclude that $h_x(\alpha) \geq h(\alpha)$ verifying the theorem in the case that $x \leq 1$.

Next, we assume that $x > 1$. Let N be a positive integer and select $\beta \in \overline{\mathbb{Q}}^\times$ such that $\beta^N = \alpha$. Therefore, we have that

$$h_x(\alpha) \leq \left(\sum_{n=1}^N h(\beta)^x \right)^{1/x} = (Nh(\beta)^x)^{1/x} = N^{1/x} \cdot h(\beta).$$

Then using (6.1) we obtain that $h(\alpha) = N \cdot h(\beta)$ which yields

$$(6.2) \quad h_x(\alpha) \leq N^{\frac{1}{x}-1} \cdot h(\alpha).$$

Since $x > 1$, the right hand side of (6.2) tends to zero as $N \rightarrow \infty$ completing the proof. \square

7. ACKNOWLEDGMENT

The author wishes to thank the Max-Planck-Institut für Mathematik where the majority of this research took place.

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